CS364A: Problem Set #1

Due in class on Tuesday, October 26, 2004

Instructions:

(1) You are expected to attempt all of the following problems, except for those labeled "extra credit" or "food for thought." If you cannot solve a problem to completion, explain your progress and where you got stuck.

(2) You may refer to your course notes and to general references (e.g., textbooks), but *not* to specific research papers.

(3) Collaboration on this homework is encouraged. However, your write-up must be your own, and you must list the names of your collaborators on the front page.

Problem 1

Recall from class our definition of a flow at Nash equilibrium:

Definition 1 A flow f is *at Nash equilibrium* in a network G with cost functions c if for all s-t paths $P_1, P_2 \in \mathcal{P}$ with $f_{P_1} > 0$,

$$c_{P_1}(f) \leq c_{P_2}(f).$$

In September 30th's lecture, several students suggested the following (arguably more intuitive) definition:

Definition 2 A flow f is *at Nash equilibrium* in a network G with cost functions c if for all s-t paths $P_1, P_2 \in \mathcal{P}$ with $f_{P_1} > 0$ and all $\delta \in (0, f_{P_1}]$,

$$c_{P_1}(f) \leq c_{P_2}(\tilde{f}),$$

where $\tilde{f}$ is defined by

$$\tilde{f}_P = \begin{cases} f_P - \delta & \text{if } P = P_1 \\ f_P + \delta & \text{if } P = P_2 \\ f_P & \text{if } P \notin \{P_1, P_2\}. \end{cases}$$

Prove that these two definitions are equivalent: whenever a flow f satisfies Definition 1 it also satisfies Definition 2, and conversely.

[Hint: Remember that cost functions are continuous and nondecreasing.]

Problem 2

Recall that in class we proved a theorem stating that upper-bounding the price of anarchy with respect to a set $\mathcal{C}$ of allowable semiconvex cost functions that includes all of the constant functions reduces to upper-bounding the price of anarchy just in Pigou-like networks, where a Pigou-like network has two vertices, two edges, and one edge has a constant cost function.

(a) For a positive integer $d \geq 1$, let $\mathcal{C}_d$ denote the set $\{ax^i : a \geq 0, i \in \{0, 1, \ldots, d\}\}$ of monomials with a nonnegative coefficient and degree at most d. What is the largest price of anarchy possible in a Pigou-like network with cost functions in $\mathcal{C}_d$?

(b) For a positive integer $d \geq 1$, let $\mathcal{C}_d$ denote the set $\{\sum_{i=0}^{d} a_i x^i \,:\, a_0, \ldots, a_d \geq 0\}$ of polynomials with nonnegative coefficients and degree at most d. What is the largest price of anarchy possible in a Pigou-like network with cost functions in $\mathcal{C}_d$?

[Hint: consider graph transformations that might reduce this calculation to that of part (a).]

(c) **(Extra credit)** What if $\mathcal{C}_d$ is the set of nondecreasing, nonnegative polynomials with degree at most d? [Note: such polynomials may have some negative coefficients.]

Problem 3

Recall that in class we proved the following "bicriteria bound."

Theorem 3 *If f is at Nash equilibrium in a network G with cost functions c and a traffic rate r, and f^* is feasible in the same network at the traffic rate $2r$, then*

$$C(f) \leq C(f^*).$$

Assuming this result, show the following: if f^* is feasible in a network G with cost functions c and f is at Nash equilibrium in the same network with cost functions $\tilde{c}$, where $\tilde{c}_e(x)$ is defined as $c_e(x/2)/2$, then

$$\tilde{C}(f) \leq C(f^*),$$

where $\tilde{C}(f) = \sum_{P \in \mathcal{P}} f_P \tilde{c}_P(f)$ and $C(f^*) = \sum_{P \in \mathcal{P}} f_P^* c_P(f^*)$.

Problem 4

Recall that in class, we have always assumed that networks are "single-commodity"—that all traffic begins at a common source vertex and travels to a common destination vertex.

(a) Extend the definition of flows and of flows at Nash equilibrium to *multicommodity networks*, which can possess multiple source and destination vertices.

(b) State and prove a generalization of the bicriteria bound above (Theorem 3) for multicommodity networks.

(c) **(Extra credit)** Show that for every set $\mathcal{C}$ of semiconvex cost functions that contains the constant functions, the largest-possible price of anarchy in multicommodity networks with cost functions in $\mathcal{C}$ is no larger than that in single-commodity networks with cost functions in $\mathcal{C}$.

(d) **(Food for thought)** Do you think that the result in part (c) still holds if:

(i) the cost functions in $\mathcal{C}$ need not be semiconvex?

(ii) $\mathcal{C}$ need not contain all of the constant functions?